Examples of surface integral

Monday, April 12, 2021 11:28 AM

<u>Recall</u>:

$$\iint_S f \, dS = \iint_D f(r(u,v))\, |r_u \times r_v| \, dA$$

(f: scalar function)

S is parametrized by $r(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$(u,v) \in D$

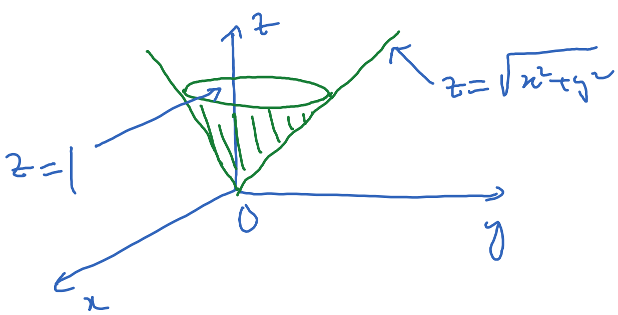
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(r(u,v)) \cdot (\pm r_u \times r_v) \, dA$$

($\vec{F}$: vector field)

$\pm$: depending on which direction is called inward / outward of the surface

$$d\vec{S} = \vec{n}\, dS = \frac{r_u \times r_v}{|r_u \times r_v|} |r_u \times r_v| \, dA$$

<u>Ex</u>



$$\iint_S xz \, dS = ?$$

Parametrize the surface:

$$\begin{cases} x = x \\ y = y \\ z = \sqrt{x^2+y^2} \end{cases} \qquad (x,y) \in D$$

$$\left.\begin{array}{l} r_x = \left\langle 1, 0, \dfrac{x}{\sqrt{x^2+y^2}} \right\rangle \\ \\ r_y = \left\langle 0, 1, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle \end{array}\right\} \ r_x \times r_y \text{ is somewhat complicated.}$$

Try:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = \sqrt{x^2+y^2} = \rho \end{cases} \qquad \underbrace{\begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \end{array}}_{D}$$

$$r_\rho = \langle \cos\theta, \sin\theta, 1 \rangle$$

$$r_\theta = \langle -\rho\sin\theta, \rho\cos\theta, 0 \rangle$$

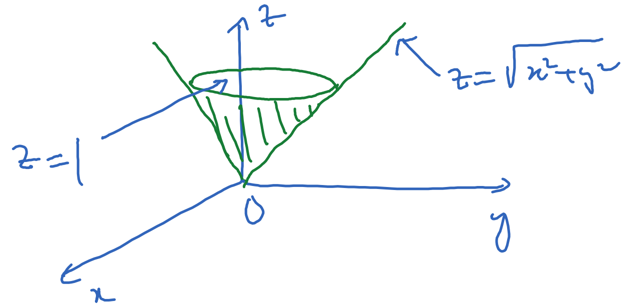
$$r_\rho \times r_\theta = \langle -\rho\cos\theta, -\rho\sin\theta, \rho \rangle$$

$$|r_\rho \times r_\theta| = \sqrt{\rho^2(\cos^2 + \sin^2\theta) + \rho^2} = \rho\sqrt{2}.$$

$$\iint_S xz\, dS = \iint_D \rho\cos\theta\, \rho\, \rho\sqrt{2}\, dA = \int_0^1 \int_0^{2\pi} \rho^3 \sqrt{2}\cos\theta\, d\theta\, d\rho$$

$$= 0$$

Ex



$$\iint_S \langle x,0,0\rangle \cdot d\vec{S} = ?$$

S ← oriented outward

normal vector $r_\rho \times r_\theta = \langle -\rho\cos\theta, -\rho\sin\theta, \rho\rangle$ ← pointing inside the cone

The normal vector pointing outside the cone is

$$-r_\rho \times r_\theta = \langle \rho\cos\theta, \rho\sin\theta, -\rho\rangle$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \underbrace{\langle \rho\cos\theta, 0, 0\rangle}_{\langle x,0,0\rangle} \cdot \underbrace{\langle \rho\cos\theta, \rho\sin\theta, -\rho\rangle}_{-r_\rho \times r_\theta}\, dA$$

$$= \int_0^1\int_0^{2\pi} \rho^2\cos^2\theta \, d\theta\, d\rho \quad = \cdots\cdots$$